

# On non-existence of full exceptional collections on some relative flags

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**Abstract.** In this short note we show that certain relative flags cannot have full exceptional collections. We also prove that some of these flags are categorically representable in dimension zero if and only if they admit a full exceptional collection. As a consequence, these flags are representable in dimension zero if and only if they have  $k$ -rational points.

## CONTENTS

1. Introduction	1
2. Generalized Brauer–Severi varieties and twisted quadrics	2
3. Exceptional collections and semiorthogonal decompositions	2
4. Recollections on noncommutative motives	4
5. Proof of Theorem 1.1	5
6. Proof of Theorem 1.2	6
References	7

## 1. INTRODUCTION

Among others, in [19] we proved that non-trivial twisted flags of classical type cannot have full exceptional collections. In view of this fact it is also natural to ask what happens for twisted forms of relative flags or for relative flags over bases which do not have full exceptional collections. In this context we prove the following theorem which is certainly known to the experts but which we still want to prove as we could not find a reference.

**Theorem 1.1.** *Let  $Z$  be a non-trivial twisted flag of type  $A_n, B_n, C_n$  or  $D_n$  ( $n \neq 4$ ) over a field  $k$  and  $\pi: X \rightarrow Z$  a flat and proper morphism with  $X$  a smooth projective  $k$ -scheme. Assume there is a semiorthogonal decomposition  $D^b(X) = \langle \pi^* D^b(Z) \otimes \mathcal{E}_1, \dots, \pi^* D^b(Z) \otimes \mathcal{E}_r \rangle$  with  $D^b(Z)$  being equivalent to  $\pi^* D^b(Z) \otimes \mathcal{E}_i$  via  $\pi^*(-) \otimes \mathcal{E}_i$ . Then  $X$  cannot have a full exceptional collection.*

Note that although the  $X$  from Theorem 1.1 cannot have a full exceptional collection it can have a tilting bundle (see [20] for examples).

For the next result, let  $G = \mathrm{PSO}_n$  be over  $k$  with  $n$  even and  $\mathrm{char}(k) \neq 2$ . Given a 1-cocycle  $\gamma: \mathrm{Gal}(k^s|k) \rightarrow \mathrm{PSO}_n(k^s)$  we get a twisted form of a quadric  ${}_\gamma Q$  and a central simple  $k$ -algebra  $(A, \sigma)$  of degree  $n$  with involution associated to  $\gamma$  (see [13]). Note that  ${}_\gamma Q$  is isomorphic to the involution variety  $\mathrm{IV}(A, \sigma)$  of Section 2. For any splitting field  $L$  of  $A$ , the variety  ${}_\gamma Q \otimes_k L$  is isomorphic to a smooth quadric in  $\mathbb{P}_L^{n-1}$ .

Note that the (generalized) Brauer–Severi varieties are obtained as quotients of  $G = \mathrm{PGL}_n$  by a certain parabolic subgroup  $P$  and by twisting with a 1-cocycle  $\gamma: \mathrm{Gal}(k^s|k) \rightarrow \mathrm{PGL}_n(k^s)$ . In Section 6 we show the following:

**Theorem 1.2.** *Let  $Z$  be either a Brauer–Severi variety over an arbitrary field  $k$ , a generalized Brauer–Severi variety over a field of characteristic zero or a smooth twisted quadric*

from above and  $\pi: X \rightarrow Z$  a flat and proper morphism where  $X$  is a smooth projective  $k$ -scheme. Assume there is a semiorthogonal decomposition  $D^b(X) = \langle \pi^* D^b(Z) \otimes \mathcal{E}_1, \dots, \pi^* D^b(Z) \otimes \mathcal{E}_r \rangle$  as in Theorem 1.1. Then  $X$  is categorically representable in dimension zero if and only if the 1-cocycle defining  $Z$  is trivial.

Theorem 1.2 has the following simple consequences.

**Corollary 1.3.** *Under the assumptions of Theorem 1.1,  $X$  is categorically representable in dimension zero if and only if it admits a full exceptional collection.*

**Corollary 1.4.** *Let  $Z$  be a Brauer–Severi variety over an arbitrary field  $k$  or a smooth twisted quadric as above associated to an isotropic involution algebra and  $\mathcal{E}$  a vector bundle on  $Z$ . Then  $\mathbb{P}_Z(\mathcal{E})$  and  $\text{Grass}_Z(d, \mathcal{E})$  are categorically representable in dimension zero if and only if they admit a  $k$ -rational point.*

## 2. GENERALIZED BRAUER–SEVERI VARIETIES AND TWISTED QUADRICS

A Brauer–Severi variety of dimension  $n$  is a scheme  $X$  of finite type over  $k$  such that  $X \otimes_k L \simeq \mathbb{P}^n$  for a finite field extension  $k \subset L$ . A field extension  $k \subset L$  for which  $X \otimes_k L \simeq \mathbb{P}^n$  is called *splitting field* of  $X$ . Clearly,  $k^s$  and  $\bar{k}$  are splitting fields for any Brauer–Severi variety. In fact, every Brauer–Severi variety always splits over a finite Galois extension of  $k$ . It follows from descent theory that  $X$  is projective, integral and smooth over  $k$ . Via non-commutative Galois cohomology, Brauer–Severi varieties of dimension  $n$  are in one-to-one correspondence with central simple algebras  $A$  of degree  $n + 1$ . For details and proofs on all mentioned facts we refer to [2] and [10].

To a central simple  $k$ -algebra  $A$  one can also associate twisted forms of Grassmannians. Let  $A$  be of degree  $n$  and  $1 \leq d \leq n$ . Consider the subset of  $\text{Grass}_k(d \cdot n, A)$  consisting of those subspaces of  $A$  that are left ideals  $I$  of dimension  $d \cdot n$ . This subset can be given the structure of a projective variety which turns out to be a generalized Brauer–Severi variety. It is denoted by  $\text{BS}(d, A)$ . After base change to some splitting field  $E$  of  $A$  the variety  $\text{BS}(d, A)$  becomes isomorphic to  $\text{Grass}_E(d, n)$ . If  $d = 1$  the generalized Brauer–Severi variety is the Brauer–Severi variety associated to  $A$ . Note that  $\text{BS}(d, A)$  is a Fano variety. For details see [6].

Finally, to a central simple algebra  $A$  of degree  $n$  with involution  $\sigma$  of the first kind over a field  $k$  of  $\text{char}(k) \neq 2$  one can associate the *involution variety*  $\text{IV}(A, \sigma)$ . This variety can be described as the variety of  $n$ -dimensional right ideals  $I$  of  $A$  such that  $\sigma(I) \cdot I = 0$ . If  $A$  is split so  $(A, \sigma) \simeq (M_n(k), q^*)$ , where  $q^*$  is the adjoint involution defined by a quadratic form  $q$  one has  $\text{IV}(A, \sigma) \simeq V(q) \subset \mathbb{P}_k^{n-1}$ . Here  $V(q)$  is the quadric determined by  $q$ . By construction such an involution variety  $\text{IV}(A, \sigma)$  becomes a quadric in  $\mathbb{P}_L^{n-1}$  after base change to some splitting field  $L$  of  $A$ . In this way the involution variety is a twisted form of a smooth quadric as defined before. Recall from [27] that a splitting field  $L$  of  $A$  is called *isotropically* if  $(A, \sigma) \otimes_k L \simeq (M_n(L), q^*)$  with  $q$  an isotropic quadratic form over  $L$ . For details on the construction and further properties on involution varieties and the corresponding algebras we refer to [27].

## 3. EXCEPTIONAL COLLECTIONS AND SEMIORTHOGONAL DECOMPOSITIONS

Let  $\mathcal{D}$  be a triangulated category and  $\mathcal{C}$  a triangulated subcategory. The subcategory  $\mathcal{C}$  is called *thick* if it is closed under isomorphisms and direct summands. For a subset  $A$  of objects of  $\mathcal{D}$  we denote by  $\langle A \rangle$  the smallest full thick subcategory of  $\mathcal{D}$  containing the elements of  $A$ . Furthermore, we define  $A^\perp$  to be the subcategory of  $\mathcal{D}$  consisting of all objects  $M$  such that  $\text{Hom}_{\mathcal{D}}(E[i], M) = 0$  for all  $i \in \mathbb{Z}$  and all elements  $E$  of  $A$ . We say that  $A$  *generates*  $\mathcal{D}$  if  $A^\perp = 0$ . Now assume  $\mathcal{D}$  admits arbitrary direct sums. An object  $B$  is called *compact* if  $\text{Hom}_{\mathcal{D}}(B, -)$  commutes with direct sums. Denoting by  $\mathcal{D}^c$  the subcategory of compact objects we say that  $\mathcal{D}$  is *compactly generated* if the objects of  $\mathcal{D}^c$  generate  $\mathcal{D}$ . One has the following important theorem (see [8], Theorem 2.1.2).

**Theorem 3.1.** *Let  $\mathcal{D}$  be a compactly generated triangulated category. Then a set of objects  $A \subset \mathcal{D}^c$  generates  $\mathcal{D}$  if and only if  $\langle A \rangle = \mathcal{D}^c$ .*

For a smooth projective scheme  $X$  over  $k$ , we denote by  $D(\text{Qcoh}(X))$  the derived category of quasicoherent sheaves on  $X$ . The bounded derived category of coherent sheaves is denoted by  $D^b(X)$ . Note that  $D(\text{Qcoh}(X))$  is compactly generated with compact objects being all of  $D^b(X)$ . For details on generating see [8].

**Definition 3.2.** Let  $A$  be a division algebra over  $k$ , not necessarily central. An object  $\mathcal{E} \in D^b(X)$  is called *w-exceptional* if  $\text{End}(\mathcal{E}) = A$  and  $\text{Hom}(\mathcal{E}, \mathcal{E}[r]) = 0$  for  $r \neq 0$ . If  $A = k$  the object is called *exceptional*. If  $A$  is a separable  $k$ -algebra, the object  $\mathcal{E}$  is called *separable-exceptional*.

**Definition 3.3.** A totally ordered set  $\{\mathcal{E}_1, \dots, \mathcal{E}_n\}$  of w-exceptional (resp. separable-exceptional) objects on  $X$  is called an *w-exceptional collection* (resp. *separable-exceptional collection*) if  $\text{Hom}(\mathcal{E}_i, \mathcal{E}_j[r]) = 0$  for all integers  $r$  whenever  $i > j$ . An w-exceptional (resp. separable-exceptional) collection is *full* if  $\langle \{\mathcal{E}_1, \dots, \mathcal{E}_n\} \rangle = D^b(X)$  and *strong* if  $\text{Hom}(\mathcal{E}_i, \mathcal{E}_j[r]) = 0$  whenever  $r \neq 0$ . If the set  $\{\mathcal{E}_1, \dots, \mathcal{E}_n\}$  consists of exceptional objects it is called *exceptional collection*.

Notice that the direct sum of objects forming a full strong w-exceptional (resp. separable-exceptional) collection is a tilting object.

**Example 3.4.** Let  $\mathbb{P}^n$  be the projective space and consider the ordered collection of invertible sheaves  $\{\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n)\}$ . In [4] Beilinson showed that this is a full strong exceptional collection.

**Example 3.5.** Let  $X = \mathbb{P}^1 \times \mathbb{P}^1$ . Then  $\{\mathcal{O}_X, \mathcal{O}_X(1, 0), \mathcal{O}_X(0, 1), \mathcal{O}_X(1, 1)\}$  is a full strong exceptional collection on  $X$ . We use the notion  $\mathcal{O}_X(i, j)$  for  $\mathcal{O}(i) \boxtimes \mathcal{O}(j)$ .

A generalization of the notion of a full w-exceptional collection is that of a semiorthogonal decomposition of  $D^b(X)$ . Recall that a full triangulated subcategory  $\mathcal{D}$  of  $D^b(X)$  is called *admissible* if the inclusion  $\mathcal{D} \hookrightarrow D^b(X)$  has a left and right adjoint functor.

**Definition 3.6.** Let  $X$  be a smooth projective variety over  $k$ . A sequence  $\mathcal{D}_1, \dots, \mathcal{D}_n$  of full triangulated subcategories of  $D^b(X)$  is called *semiorthogonal* if all  $\mathcal{D}_i \subset D^b(X)$  are admissible and  $\mathcal{D}_j \subset \mathcal{D}_i^\perp = \{\mathcal{F} \in D^b(X) \mid \text{Hom}(\mathcal{G}, \mathcal{F}) = 0, \forall \mathcal{G} \in \mathcal{D}_i\}$  for  $i > j$ . Such a sequence defines a *semiorthogonal decomposition* of  $D^b(X)$  if the smallest full thick subcategory containing all  $\mathcal{D}_i$  equals  $D^b(X)$ .

For a semiorthogonal decomposition we write  $D^b(X) = \langle \mathcal{D}_1, \dots, \mathcal{D}_n \rangle$ .

**Example 3.7.** Let  $\mathcal{E}_1, \dots, \mathcal{E}_n$  be a full w-exceptional collection on  $X$ . It is easy to verify that by setting  $\mathcal{D}_i = \langle \mathcal{E}_i \rangle$  one gets a semiorthogonal decomposition  $D^b(X) = \langle \mathcal{D}_1, \dots, \mathcal{D}_n \rangle$ .

**Remark 3.8.** In [19] it is proved that non-trivial twisted flags of classical type cannot have full exceptional collections. In the present note we show in Section 5 that the same is true for certain relative flags. So instead of seeking full exceptional collections on schemes defined over arbitrary fields  $k$  one should look for full weak or separable-exceptional collections which have been proved to exist in certain cases.

For a wonderful and comprehensive overview of the theory on semiorthogonal decompositions and its relevance in algebraic geometry we refer to [14].

Recall the following definitions given in [5].

**Definition 3.9.** A  $k$ -linear triangulated category  $\mathcal{T}$  is *representable in dimension  $m$*  if it admits a semiorthogonal decomposition  $\mathcal{T} = \langle \mathcal{T}_1, \dots, \mathcal{T}_r \rangle$  and for each  $i = 1, \dots, r$  there exists a smooth projective variety  $Y_i$  over  $k$  with  $\dim(Y_i) \leq m$  such that  $\mathcal{T}_i$  is equivalent to an admissible subcategory of  $D^b(Y_i)$ .

**Definition 3.10.** Let  $X$  be a smooth projective variety over  $k$  of dimension  $n$ . We say that  $X$  is *categorically representable* in dimension  $m$  if the  $k$ -linear triangulated category  $D^b(X)$  is representable in dimension  $m$ .

#### 4. RECOLLECTIONS ON NONCOMMUTATIVE MOTIVES

We refer to [23] and [17] for a survey on noncommutative motives. Let  $\mathcal{A}$  be a small dg category. The homotopy category  $H^0(\mathcal{A})$  has the same objects as  $\mathcal{A}$  and as morphisms  $H^0(\text{Hom}_{\mathcal{A}}(x, y))$ . A source of examples is provided by schemes since the derived category of perfect complexes  $\text{perf}(X)$  of any quasi-compact quasi-separated scheme  $X$  admits a canonical dg enhancement  $\text{perf}_{dg}(X)$  (for details see [12]). Note that for smooth projective  $k$ -schemes  $X$  one has  $D^b(X) = \text{perf}(X)$ . Denote by **dgc** the category of small dg categories. The *opposite* dg category  $\mathcal{A}^{op}$  has the same objects as  $\mathcal{A}$  and  $\text{Hom}_{\mathcal{A}^{op}}(x, y) := \text{Hom}_{\mathcal{A}}(y, x)$ . A *right  $\mathcal{A}$ -module* is a dg functor  $\mathcal{A}^{op} \rightarrow C_{dg}(k)$  with values in the dg category  $C_{dg}(k)$  of complexes of  $k$ -vector spaces. We write  $C(\mathcal{A})$  for the category of right  $\mathcal{A}$ -modules. Recall from [12] that the *derived category*  $D(\mathcal{A})$  of  $\mathcal{A}$  is the localization of  $C(\mathcal{A})$  with respect to quasi-isomorphisms. A dg functor  $F: \mathcal{A} \rightarrow \mathcal{B}$  is called *derived Morita equivalence* if the restriction of scalars functor  $D(\mathcal{B}) \rightarrow D(\mathcal{A})$  is an equivalence. The *tensor product*  $\mathcal{A} \otimes \mathcal{B}$  of two dg categories is defined as follows: the set of objects is the cartesian product of the sets of objects in  $\mathcal{A}$  and  $\mathcal{B}$  and  $\text{Hom}_{\mathcal{A} \otimes \mathcal{B}}((x, w), (y, z)) := \text{Hom}_{\mathcal{A}}(x, y) \otimes \text{Hom}_{\mathcal{B}}(w, z)$  (see [12]). Given two dg categories  $\mathcal{A}$  and  $\mathcal{B}$ , let  $\text{rep}(\mathcal{A}, \mathcal{B})$  be the full triangulated subcategory of  $D(\mathcal{A}^{op} \otimes \mathcal{B})$  consisting of those  $\mathcal{A} - \mathcal{B}$ -bimodules  $M$  such that  $M(x, -)$  is a compact object of  $D(\mathcal{B})$  for every object  $x \in \mathcal{A}$ . Now there is an additive symmetric monoidal category  $\text{Hmo}_0$  with objects being small dg categories and morphisms being

$$\text{Hom}_{\text{Hmo}_0}(\mathcal{A}, \mathcal{B}) \simeq K_0(\text{rep}(\mathcal{A}, \mathcal{B})).$$

To any such small dg category  $\mathcal{A}$  one can associate functorially its noncommutative motive  $U(\mathcal{A})$  which takes values in  $\text{Hmo}_0$ . This functor  $U: \mathbf{dgc} \rightarrow \text{Hmo}_0$  is proved to be the *universal additive invariant* (see [23]). Recall from [26] that an additive invariant is any functor  $E: \mathbf{dgc} \rightarrow \mathcal{D}$  taking values in an additive category  $\mathcal{D}$  such that

- (i) it sends derived Morita equivalences to isomorphisms,
- (ii) for any pre-triangulated dg category  $\mathcal{A}$  admitting full pre-triangulated dg subcategories  $\mathcal{B}$  and  $\mathcal{C}$  such that  $H^0(\mathcal{A}) = \langle H^0(\mathcal{B}), H^0(\mathcal{C}) \rangle$  is a semiorthogonal decomposition, the morphism  $E(\mathcal{B}) \oplus E(\mathcal{C}) \rightarrow E(\mathcal{A})$  induced by the inclusions is an isomorphism.

Now let  $G$  split simply connected semi-simple algebraic group over the field  $k$  and  $P$  a parabolic subgroup. We denote by  $\tilde{G}$  and  $\tilde{P}$  their universal covers. For the center  $\tilde{Z} \subset \tilde{G}$  let  $\text{Ch} := \text{Hom}(\tilde{Z}, \mathbb{G}_m)$  be the character group. Furthermore, let  $R(\tilde{G})$  and  $R(\tilde{P})$  be the associated representation rings. Recall from [22], §2 that there exists a finite free  $\text{Ch}$ -homogeneous basis of  $R(\tilde{P})$  over  $R(\tilde{G})$ . Moreover, to a 1-cocycle  $\gamma: \text{Gal}(k^s|k) \rightarrow G(k^s)$  one has the *Tit's map* (see [22], §3 or [13], p.377)  $\beta_\gamma: \text{Ch} \rightarrow \text{Br}(k)$  which is a group homomorphism and assigns to each character  $\chi \in \text{Ch}$  a central simple algebra  $A_{\chi, \gamma} \in \text{Br}(k)$ , called *Tit's algebra*. If  $\rho_1, \dots, \rho_n$  is the  $\text{Ch}$ -homogeneous  $R(\tilde{G})$  basis of  $R(\tilde{P})$  we write  $\chi(i)$  for the character such that  $\rho_i \in R^{\chi(i)}(\tilde{P})$  (see [22], [13] and [18] for details). Under this notation one has the following theorem:

**Theorem 4.1** ([26], Theorem 2.1 (i)). *Let  $G$ ,  $P$  and  $\gamma$  be as above and  $E: \mathbf{dgc} \rightarrow \mathcal{D}$  an additive invariant. Then every  $\text{Ch}$ -homogeneous basis  $\rho_1, \dots, \rho_n$  of  $R(\tilde{P})$  over  $R(\tilde{G})$  give rise to an isomorphism*

$$\bigoplus_{i=1}^n E(A_{\chi(i), \gamma}) \xrightarrow{\sim} E(\text{perf}_{dg}(\gamma X)),$$

where  $A_{\chi(i), \gamma}$  are the Tits central simple algebras associated to  $\rho_i$  via  $\beta_\gamma: \text{Ch} \rightarrow \text{Br}(k)$ .

**Theorem 4.2** ([26], Theorem 3.3). *Let  $G$ ,  $P$  and  $\gamma$  as in Theorem 4.1. Then  $\bigoplus_{i=1}^n U(k) \simeq U(\text{perf}_{dg}(\gamma X))$  if and only if the Brauer classes  $[A_{\chi(i), \gamma}]$  are trivial.*

## 5. PROOF OF THEOREM 1.1

*Proof.* From the assumption that there is a semiorthogonal decomposition

$$D^b(X) = \langle \pi^* D^b(Z) \otimes \mathcal{E}_1, \dots, \pi^* D^b(Z) \otimes \mathcal{E}_r \rangle$$

we obtain from [15], Proposition 4.10 that there are pretriangulated dg categories  $\mathcal{T}_1, \dots, \mathcal{T}_r$  with  $H^0(\mathcal{T}_i) \simeq \pi^* D^b(Z) \otimes \mathcal{E}_i$ . As  $D^b(Z)$  is equivalent to  $\pi^* D^b(Z) \otimes \mathcal{E}_i$  and since  $D^b(Z)$  has a unique dg enhancement according to [16] we conclude

$$U(\text{perf}_{dg}(X)) \simeq U(\text{perf}_{dg}(Z)) \oplus \dots \oplus U(\text{perf}_{dg}(Z)).$$

According to Theorem 4.1 one has  $U(\text{perf}_{dg}(Z)) = \bigoplus U(A_i)$  for some central simple  $k$ -algebras  $A_i$ . Assuming the existence of a full exceptional collection on  $X$  we obtain

$$\begin{aligned} U(\text{perf}_{dg}(X)) &\simeq U(k) \oplus \dots \oplus U(k) \\ &\simeq \left( \bigoplus_i U(A_i) \right) \oplus \dots \oplus \left( \bigoplus_i U(A_i) \right). \end{aligned}$$

Then Theorem 4.2 implies that all  $A_i$  must split. Now see [19], Proposition 5.6 to conclude that the 1-cocycle  $\gamma$  defining the twisted flag  $Z$  must be trivial too. But this contradicts the assumption that  $Z$  is a non-trivial twisted flag.  $\square$

**Corollary 5.1.** *Let  $Z$  be as in Theorem 1.1 and  $\mathcal{E}$  a vector bundle on  $Z$ . Then  $\mathbb{P}_Z(\mathcal{E})$  cannot have a full exceptional collection.*

*Proof.* Let  $r$  be the rank of  $\mathcal{E}$ . Recall from [21] that one has a semiorthogonal decomposition

$$(1) \quad D^b(\mathbb{P}_Z(\mathcal{E})) \simeq \langle \pi^* D^b(Z) \otimes \mathcal{O}_{\mathcal{E}}, \dots, \pi^* D^b(Z) \otimes \mathcal{O}_{\mathcal{E}}(r-1) \rangle.$$

Note that this semiorthogonal decomposition also exists over arbitrary base fields  $k$  (see for instance [11], p.184). It is easy to see that the triangulated category  $D^b(Z)$  is equivalent to  $\pi^* D^b(Z) \otimes \mathcal{O}_{\mathcal{E}}(i)$  via  $\pi^*(-) \otimes \mathcal{O}_{\mathcal{E}}(i)$ . Now Theorem 1.1 yields the assertion.  $\square$

**Corollary 5.2.** *Let  $Z$  be as before and assume the base field  $k$  is of characteristic zero. Furthermore, let  $\mathcal{E}$  be a vector bundle on  $Z$ . Then  $\text{Grass}_Z(d, \mathcal{E})$  cannot have a full exceptional collection.*

*Proof.* Let  $r+1$  be the rank of  $\mathcal{E}$  and denote by  $\mathcal{R}$  the tautological subbundle of rank  $d$  in  $\pi^*(\mathcal{E})$ . Moreover, let  $P$  be the set of partitions  $\lambda = (\lambda_1, \dots, \lambda_d)$  with  $0 \leq \lambda_d \leq \dots \leq \lambda_1 \leq r+1-d$ . One can choose a total order  $\prec$  on  $P$  such that  $\lambda \prec \mu$  means that the Young diagram of  $\lambda$  is not contained in that of  $\mu$ . Recall from [3] that one has a semiorthogonal decomposition

$$D^b(\text{Grass}_Z(d, \mathcal{E})) = \langle \dots \pi^* D^b(Z) \otimes \Sigma^\lambda(\mathcal{R}), \dots, \pi^* D^b(Z) \otimes \Sigma^\mu(\mathcal{R}), \dots \rangle$$

where  $\lambda \prec \mu$ . It is easy to see that  $D^b(Z)$  is equivalent to  $\pi^* D^b(Z) \otimes \Sigma^\lambda(\mathcal{R})$  via  $\pi^*(-) \otimes \Sigma^\lambda(\mathcal{R})$ . Indeed, this follows from adjunction of  $\pi^*$  and  $\pi_*$ , projection formula (see [11]) and the relative version of the Borel–Weil–Bott Theorem (see [9], Theorem 5.1). Then Theorem 1.1 yields the assertion.  $\square$

## 6. PROOF OF THEOREM 1.2

We need the following well-known lemma.

**Lemma 6.1.** *Let  $\pi: X \rightarrow Z$  be a flat proper morphism between smooth projective  $k$ -schemes. Assume the existence of a semiorthogonal decomposition  $D^b(X) = \langle \pi^* D^b(Z) \otimes \mathcal{E}_1, \dots, \pi^* D^b(Z) \otimes \mathcal{E}_r \rangle$  with  $D^b(Z)$  being equivalent to  $\pi^* D^b(Z) \otimes \mathcal{E}_i$  via  $\pi^*(-) \otimes \mathcal{E}_i$ . Assume furthermore that  $\{\mathcal{A}_1, \dots, \mathcal{A}_m\}$  is a full exceptional collection for  $Z$ . Then the ordered set*

$$S = \{\pi^*(\mathcal{A}_1) \otimes \mathcal{E}_1, \dots, \pi^*(\mathcal{A}_m) \otimes \mathcal{E}_1, \dots, \pi^*(\mathcal{A}_1) \otimes \mathcal{E}_r, \dots, \pi^*(\mathcal{A}_m) \otimes \mathcal{E}_r\}$$

*is a full exceptional collection for  $X$ .*

*Proof.* Since  $\text{Hom}(\pi^*(\mathcal{A}_l) \otimes \mathcal{E}_i, \pi^*(\mathcal{A}_l) \otimes \mathcal{E}_i) \simeq \text{Hom}(\mathcal{A}_l, \mathcal{A}_l) = k$  we see that all the elements of  $S$  are exceptional. From the assumption that  $D^b(X) = \langle \pi^* D^b(Z) \otimes \mathcal{E}_1, \dots, \pi^* D^b(Z) \otimes \mathcal{E}_r \rangle$  is a semiorthogonal decomposition it is easy to conclude that  $S$  generates  $D^b(X)$  and that furthermore  $\text{Hom}(\pi^*(\mathcal{A}_l) \otimes \mathcal{E}_i, \pi^*(\mathcal{A}_q) \otimes \mathcal{E}_j[p]) = 0$  for all  $p \in \mathbb{Z}$  whenever  $i > j$ .  $\square$

For the proof of Theorem 1.2 we need some notations. Denote by  $\text{NChow}(k)$  the category of non-commutative Chow motives (see [24] for details). Now let  $\text{CSep}(k)$  be the full subcategory of  $\text{NChow}(k)$  consisting of objects of the form  $U(A)$  with  $A$  a commutative separable  $k$ -algebra. Analogously,  $\text{Sep}(k)$  denotes the full subcategory of  $\text{NChow}(k)$  consisting of objects  $U(A)$  with  $A$  a separable  $k$ -algebra. And finally, we write  $\text{CSA}(k)$  for the full subcategory of  $\text{Sep}(k)$  consisting of  $U(A)$  with  $A$  being a central simple  $k$ -algebra.

*Proof.* (of Theorem 1.2)

Assume  $X$  is categorically representable in dimension zero. From [1], Lemma 1.19 it follows

$$D^b(X) = \langle D^b(K_1), \dots, D^b(K_s) \rangle,$$

where  $K_1, \dots, K_r$  are étale  $k$ -algebras. From the assumption we then obtain

$$\langle \pi^* D^b(Z) \otimes \mathcal{E}_1, \dots, \pi^* D^b(Z) \otimes \mathcal{E}_r \rangle = \langle D^b(K_1), \dots, D^b(K_s) \rangle.$$

From [15], Proposition 4.10 we obtain pretriangulated dg categories  $\mathcal{T}_1, \dots, \mathcal{T}_r$  such that  $H^0(\mathcal{T}_i) \simeq \pi^* D^b(Z) \otimes \mathcal{E}_i$ . As  $D^b(Z) \simeq \pi^* D^b(Z) \otimes \mathcal{E}_i$  for  $1 \leq i \leq r$  and since the dg enhancement of  $D^b(Z)$  is unique (see [16]) we get  $U(\text{perf}_{dg}(Z)) \simeq U(\mathcal{T}_i)$  for all  $1 \leq i \leq r$ . The noncommutative motives of the left hand respectively right hand side therefore satisfy

$$U(\text{perf}_{dg}(Z)) \oplus \dots \oplus U(\text{perf}_{dg}(Z)) \simeq U(K_1) \oplus \dots \oplus U(K_s).$$

Assuming that  $Z$  is either a Brauer–Severi variety over an arbitrary field  $k$  or a generalized Brauer–Severi variety over a field of characteristic zero, we conclude from Theorem 4.1 that  $U(\text{perf}_{dg}(Z)) = \bigoplus_j U(A_j)$  where  $A_j$  are central simple  $k$ -algebras. Thus

$$\bigoplus_j U(A_j)^{\oplus r} \simeq U(K_1) \oplus \dots \oplus U(K_s).$$

Recall from [25] that one has the following 2-cartesian square of categories (see [25], (2.16) and Corollary 2.13)

$$\begin{array}{ccc} \{U(k)^{\oplus n} \mid n \geq 0\} & \longrightarrow & \text{CSA}(k)^{\oplus} \\ \downarrow & & \downarrow \\ \text{CSep}(k) & \longrightarrow & \text{Sep}(k) \end{array}$$

which gives an equivalence of categories  $\{U(k)^{\oplus n} \mid n \geq 0\} \simeq \text{CSA}(k)^{\oplus} \times_{\text{Sep}(k)} \text{CSep}(k)$ . Here  $\text{CSA}(k)^{\oplus}$  denotes the closure of  $\text{CSA}(k)$  under finite direct sums. Now the above 2-cartesian square, or more precise the universal property of fiber products, implies that

such an isomorphism is possible if only if  $K_1 = \dots = K_s = k$  and all  $A_j$  are split. If  $Z$  is a smooth twisted quadric, we conclude from [7] that  $Z$  has a semiorthogonal decomposition

$$D^b(Z) = \langle D^b(k), D^b(A), \dots, D^b(A), D^b(k), D^b(A), D^b(C_0^-(A, \sigma)), D^b(C_0^+(A, \sigma)) \rangle.$$

Here  $k, A, C_0^-(A, \sigma)$  and  $C_0^+(A, \sigma)$  are the minimal Tit's algebras of  $Z$ . Hence we get

$$\left( U(k) \oplus U(A) \oplus \dots \oplus U(C_0^-(A, \sigma)) \oplus U(C_0^+(A, \sigma)) \right)^{\oplus r} \simeq U(K_1) \oplus \dots \oplus U(K_r).$$

Note that this isomorphism follows also directly from [26], Example 3.11. Again the above 2-cartesian square implies  $K_1 = \dots = K_s = k$  and  $A, C_0^-(A, \sigma)$  and  $C_0^+(A, \sigma)$  are split. This in particular implies that the 1-cocycle which determines  $Z$  must be trivial.

Now assume the 1-cocycle which determines  $Z$  is trivial. Then see [7] to conclude that  $Z$  has a full exceptional collection. Then Lemma 6.1 provides us with a full exceptional collection for  $X$ . Again [1], Lemma 1.19 immediately implies that  $X$  is categorically representable in dimension zero.  $\square$

*Proof.* (of Corollary 1.3)

We assume that  $X$  is categorically representable in dimension zero. Theorem 1.2 implies that the 1-cocycle which defines  $Z$  must be trivial. Now see [7] to conclude that  $Z$  admits a full exceptional collection. Lemma 6.1 gives us a full exceptional collection for  $X$ . To the contrary, if  $X$  admits a full exceptional collection then Lemma 1.19 of [1] immediately implies that  $X$  is categorically representable in dimension zero.  $\square$

*Proof.* (of Corollary 1.4)

We prove the statement only for  $\mathbb{P}_Z(\mathcal{E})$  as the other case can be shown analogously. If  $\mathbb{P}_Z(\mathcal{E})$  has a  $k$ -rational point, then so does  $Z$ . Now [19], Theorem 6.3, Corollary 6.4 and Proposition 6.9 imply that  $Z$  admits a full exceptional collection. Lemma 6.1 provides us with a full exceptional collection for  $X$  and Lemma 1.19 of [1] shows that  $X$  is categorically representable in dimension zero.

If  $X$  is categorically representable in dimension zero, Theorem 1.2 implies that the 1-cocycle defining  $Z$  must be trivial. Hence  $Z$  is a projective space or a smooth isotropic quadric and admits therefore a  $k$ -rational point  $z_0 \in Z$ . Let  $\pi^{-1}(z_0) \subset \mathbb{P}_Z(\mathcal{E})$  be the fiber. Note that  $\pi^{-1}(z_0) \simeq \mathbb{P}_k^m$  where  $m+1$  is the rank of  $\mathcal{E}$ . As  $\mathbb{P}_k^m$  has a  $k$ -rational point, we also have one on  $\mathbb{P}_Z(\mathcal{E})$ . This completes the proof.  $\square$

**Example 6.2.** Let  $Z$  be as in Corollary 5.1 or 5.2 and  $\mathcal{E}$  a vector bundle on  $Z$ . Then  $\mathbb{P}_Z(\mathcal{E})$  respectively  $\text{Grass}_Z(d, \mathcal{E})$  is representable in dimension zero if and only if it admits a full exceptional collection if and only if it has a  $k$ -rational point.

**Remark 6.3.** The results in [19] show that Theorems 1.1, 1.2 and thus Corollaries 1.3 and 1.4 also hold if  $Z$  is the finite product of the considered varieties.

## REFERENCES

- [1] A. Auel and M. Bernardara: Semiorthogonal decompositions and birational geometry of del Pezzo surfaces over arbitrary fields. arXiv:1511.07576v1 [math.AG] (2015).
- [2] M. Artin: Brauer-Severi varieties. Brauer groups in ring theory and algebraic geometry, Lecture Notes in Math. 917, Notes by A. Verschoren, Berlin, New York: Springer-Verlag (1982), 194-210.
- [3] S. Baek: Semiorthogonal for twisted Grassmannians. arXiv:1205.1175v1 [math.AG] (2012).
- [4] A.A. Beilinson: Coherent sheaves on  $\mathbb{P}^n$  and problems in linear algebra. Funktsional. Anal. i Prilozhen. Vol. 12 (1978), 68-69.
- [5] M. Bernardara and M. Bolognesi: Categorical representability and intermediate Jacobians of Fano threefolds. EMS Ser. Congr. Rep., Eur. Math. Soc. (2013), 1-10.
- [6] A. Blanchet: Function fields of generalized Brauer-Severi varieties. Comm. Algebra. Vol. 19 (1991), 97-118.
- [7] M. Blunk: A derived equivalence for some twisted projective homogeneous varieties. arXiv:1204.0537v1 [math.AG] (2012).

- [8] A.I. Bondal and M. Van den Bergh: Generators and representability of functors in commutative and noncommutative geometry. *Mosc. Math. J.* Vol. 3 (2003), 1-36.
- [9] A. Dhillon, N. Nemire and Y. Yan: Pushforwards of tilting sheaves. [arXiv:1509.02170v1 \[math.AG\]](#) (2015).
- [10] P. Gille and T. Szamuely: Central simple algebras and Galois cohomology. *Cambridge Studies in advanced Mathematics*. 101. Cambridge University Press. (2006)
- [11] D. Huybrechts: Fourier–Mukai transforms in algebraic geometry. *Oxford Mathematical Monographs*, The Clarendon Press Oxford University Press (2006).
- [12] B. Keller: On differential graded categories. *International Congress of Mathematicians (Madrid)*, Vol. II, *Eur. Math. Soc.* (2006), 151-190.
- [13] M-A. Knus, A. Merkurjev, M. Rost and J-P. Tignol: *The Book of Involutions*. *AMS Coll. Publ.* 44, AMS, Providence, RI (1998).
- [14] A.G. Kuznetsov: Semiorthogonal decomposition in algebraic geometry. [arXiv:1404.3143v3 \[math.AG\]](#) (2015).
- [15] A.G. Kuznetsov and V. Lunts: Categorical resolution of irrational singularities. *International Mathematical Research Notices* No. 13 (2015), 4536-4625.
- [16] V. Lunts and D.O. Orlov: Uniqueness of enhancement for triangulated categories. *J. Amer. Math. Soc.* Vol. 23 (2010), 853-908.
- [17] M. Marcolli and G. Tabuada: Noncommutative motives and their applications. [arXiv:1311.2867v2 \[math.AG\]](#) (2013).
- [18] A.S. Merkurjev, I.A. Panin and A.R. Wadsworth: Index reduction formulas for twisted flag varieties I. *K-theory* Vol. 10 (1996), 517-696.
- [19] S. Novaković: Non-existence of exceptional collections on twisted flags and categorical representability via noncommutative motives. [arXiv:1607.01043v1 \[math.AG\]](#) (2016).
- [20] S. Novaković: Tilting objects on twisted forms of some relative flag varieties. [arXiv:1503.05542v1 \[math.AG\]](#) (2015).
- [21] D.O. Orlov: Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Math. USSR Izv.* Vol. 38 (1993), 133-141.
- [22] I.A. Panin: On the algebraic K-theory of twisted flag varieties. *K-Theory* Vol. 8 (1994), 541-585.
- [23] G. Tabuada: A guided tour through the garden of noncommutative motives. *Clay Math. Proc.* 16 (2012), 259-276.
- [24] G. Tabuada: Chow motives versus noncommutative motives. *J. Noncommut. Geom.* Vol. 7 (2013), 767-786.
- [25] G. Tabuada and M. Van den Bergh: Noncommutative motives of separable algebras. [arXiv:1411.7970](#) (2014).
- [26] G. Tabuada: Additive invariants of toric and twisted projective homogeneous varieties via noncommutative motives. *J. Algebra* Vol. 417 (2014), 1538.
- [27] D. Tao: A Variety Associated to an Algebra with Involution. *J. Algebra* Vol. 168 (1994), 479-520.

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